Qualifying Exam—Numerical ODE 04/29/21

Show intermediate results at all steps!

1. Consider the following homogeneous ODE with smooth coefficient $f(\cdot)$:

$$\frac{du(t)}{dt} = f(u). \tag{1}$$

Show that the Local Truncation Error of the following midpoint method is of order $O(k^2)$:

$$U^{n+1} = U^n + kf\left(U^n + \frac{1}{2}kf(U^n)\right).$$

2. The Asymptotic stability of a numerical method can studied through the Harmonic oscillator

$$\frac{dz}{dt} = Az, \quad \text{where} \quad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, \quad z = (q, v)^T.$$

Derive the stability condition for the Verlet algorithm:

$$\begin{aligned} q^{n+1} &= q^n + kv^{n+1/2}, \\ v^{n+1/2} &= v^n - \frac{k}{2}\omega^2 q^n, \\ v^{n+1} &= v^{n+1/2} - \frac{k}{2}\omega^2 q^{n+1}. \end{aligned}$$

3. Prove the global convergence of the Euler scheme for equation (1):

$$U^{n+1} = U^n + kf(U^n).$$

4. Consider the Hamiltonian dynamics:

$$\frac{dq}{dt} = \nabla_p H(q,p), \quad \frac{dp}{dt} = -\nabla_q H(q,p).$$

Show that the implicit Euler-B scheme is symplectic

$$q^{n+1} = q^n + \Delta t \nabla_p H(q^n, p^{n+1}), \quad p^{n+1} = p^n - \Delta t \nabla_q H(q^n, p^{n+1}).$$

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5. Consider linear equation

$$\frac{dz}{dt} = Az, \quad \text{where} \quad A = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \quad z = (q, p)^T.$$

Derive the phase shift of the explicit Euler scheme.

6.For rigid body problem $q_i = q_{cm} + Qr_i^0,\ i = 1, 2, \dots k$, with $Q^T Q = I_3$. Give the Hamiltonian dynamics for (q_{cm}, Q) and its momentum under external potential $V_{ext}(q_{cm}, Q)$, and formulate the Rattle Scheme for simulation.